~ ICTP Summer School ~

Lecture 1

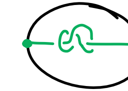
Kontsevich invariants

& configuration spaces

June 2, 2025

Let Y, X be compact oriented smooth manifolds.

homeomorphism onto the image & injective derivatives



Let $\mathrm{Emb}_\partial(Y,X)$ be the space of smooth embeddings $Y \hookrightarrow X$ that are fixed on the boundary

GOAL: understand top. properties of $\mathrm{Emb}_\partial(Y,X)$
- π_0 = conn. comp = emb./isotopy
- π_k = k-param. fam of emb. up to deformation
- H_k

For $n \geq 1$ let $\mathrm{Conf}_n X := \{(x_1,\dots,x_n) \in X^n : x_i \neq x_j \ i \neq j\} \subseteq X^n$ be the n-th configuration space of X

IDEA: define invariants of embeddings by scanning them by configurations of n points (for n large enough)

INSTANCE I: $\mathrm{Emb}_\partial(D^1, D^3)$ – Vassiliev / finite type knot theory & Gusarov-Habiro claspers

INSTANCE II: $\mathrm{Diff}_\partial(D^4)$ – Watanabe's classes   MAIN FOCUS OF THIS LECTURE SERIES

INSTANCE III: $\mathrm{Emb}_\partial(D^1, X^4)$ – graspers

For all we will see the same **pattern**:

$$\begin{tikzcd} \text{embeddings} \arrow[r, "\text{scan}"] & \frac{\text{graphs}}{\text{relations}} \\ \text{graphs} \arrow[u, "\text{explicit construction}"] & \end{tikzcd}$$

I clasper constr.

II Watanabe's constr.

configuration space integrals of Kontsevich, Bott-Taubes...

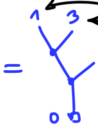
embedding calculus of Goodwillie-Weiss-Klein

(p2)

I

§ The graphs.

abstract: uni-trivalent trees with ordered set of univalent vertices with $\deg\Gamma := \#$leaves

$\Gamma =$ 

leaves.

0 called the root

! planar embedding not part of data

		Y	Ψ
deg	n	2	3
leaves	n	2	3
univ.	$n+1$	3	4
triv.	$n-1$	1	2
edges	$2n-1$	3	5

embedded: for a fixed knot $U: D^1 \hookrightarrow D^3$

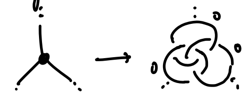
we embed $g: \Gamma \hookrightarrow D^3$ so that $g \cap U = g(\{\text{unival. vert.}\})$

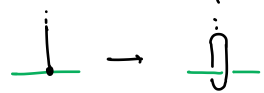
and the order on U matches their order in Γ.

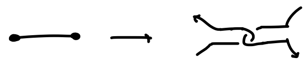
$\oplus$ all edges are framed, i.e. thickened into bands. using blackboard framing.

§ The construction.

given $g: \Gamma \hookrightarrow D^3$ on U we associate to it a framed link L_g with $2\#E$ - components as follows.

vertices: 

and  so have

edges:  or 

def. surgery on U along g is the knot $U_g \subseteq D^3_g \cong D^3$ obtained as the result of (trivial) surgery on $L_g \subseteq D^3$

That is: $D^3_{L_\Gamma} := \left(D^3 \setminus L_g\left(\bigsqcup_{6r} S^1 \times D^2\right)\right) \underset{\text{framing}}{\cup} \bigsqcup_{6r} D^2 \times S^1$, so that: $U \subseteq D^3_{L_g} \xrightarrow[\Phi_{L_g}]{\cong} D^3$ and let $U_g := \Phi_{L_g}(U)$.

(drill out L_g and glue back using the framing)

exercise. Surgery on 

$\cong S^3$.

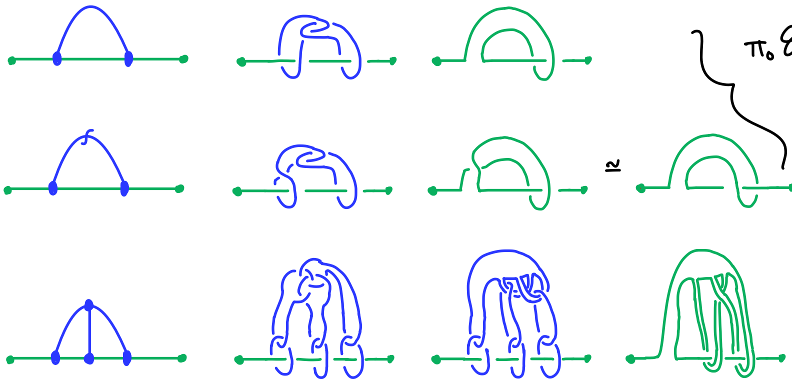
def. A knot $K: D^1 \hookrightarrow D^3$ is n-equivalent to U, written $K \sim_n U$, if $\exists g: \Gamma \hookrightarrow D^3$ on U

s.t. $K = U_g$ & $\deg\Gamma = n$.

§ examples.

n = 1

n = 2



$$\pi_0\, \mathrm{Emb}_\partial(D^1, D^3)/\sim_1 \;\cong\; \frac{\text{knots}}{\text{crossing changes}} = \{U\}$$

exercise. This is the right-handed trefoil 3_1.
Twist on one leaf edge gives fig8 knot 4_1.

$$\Rightarrow \pi_0\, \mathrm{Emb}_\partial(D^1, D^3)/\sim_2 = \{U\}$$

Moreover: $\pi_0\, \mathrm{Emb}_\partial(D^1, D^3)/\sim_3 = \mathbb{Z}$ gen. by 3_1 or 4_1.

§ THM [Kontsevich '93, additive vers. Habiro '00, Gusarov '00, Conant-Teichner '04]

$$\begin{array}{ccc} \pi_0\, \mathrm{Emb}_\partial(D^1, D^3)/\sim_{n+1} & \xrightarrow{\text{② } \exists\, Z_{<n+1}} & \bigoplus_{k=1}^{n} \mathcal{A}_k^{1,3,\mathrm{tree}} \otimes \mathbb{Q} \\ \uparrow R_{<n+1} & & \\ \bigoplus_{k=1}^{n} \mathbb{Z}[\mathrm{Tree}_k] & & \end{array}$$

① surj. homom. of abelian groups

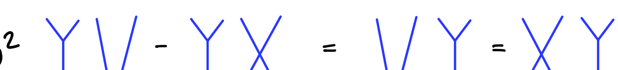
③ $U_\Gamma \longmapsto Z_k(U_\Gamma) = [\Gamma]$, $\quad [\Gamma] \mapsto U_\Gamma$

Here: $\mathcal{A}_k^{1,3,\mathrm{tree}} := \dfrac{\mathbb{Z}[\mathrm{Tree}_k]}{STU^2}$

Z_k constructed using integration over (compactified) config. spaces

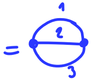
Cor. $\pi_0\, \mathrm{Emb}_\partial(D^1, D^3)/\sim_{n+1} \otimes \mathbb{Q} \cong \bigoplus_{k=1}^{n} \mathcal{A}_k^{1,3,\mathrm{tree}} \otimes \mathbb{Q}$.

STU^2



II § The graphs.

abstract: connected trivalent graphs with ordered set of edges with $\deg \Gamma := \frac{1}{2} \# V$

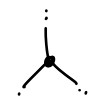
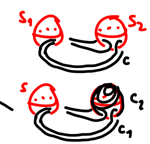
$\Gamma =$ 

$6 \deg \Gamma = 3 \# V = 2 \# E$

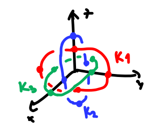
embedded: for a fixed 4-manifold X we embed $g : \Gamma \hookrightarrow X \setminus \partial X$
additionally all edges are framed, i.e. thickened into bands.

§ The construction.

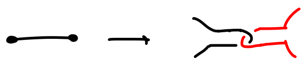
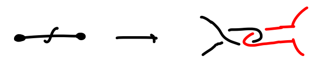
given $g : \Gamma \hookrightarrow X$ we associate to it a family of framed links $L_{g,t}$ for $t \in [0,1]^{\deg \Gamma}$ each with $2 \cdot \# E$ components ($\# E$ circles S^1, $\# E$ spheres S^2) as follows.

vertices:  $\longrightarrow$ Borr. link in 4-space 

FORMULA: $\mathbb{R}^d = \mathbb{R}^{d-p-1}_x \times \mathbb{R}^{d-q-1}_y \times \mathbb{R}^{d-r-1}_z$



$x=0$	$\frac{1}{4}\lvert y\rvert^2 + \lvert z\rvert^2 = 1$	K_1^p
$y=0$	$\frac{1}{4}\lvert z\rvert^2 + \lvert x\rvert^2 = 1$	K_2^q
$z=0$	$\frac{1}{4}\lvert x\rvert^2 + \lvert y\rvert^2 = 1$	K_3^r

edges:  or 

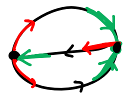
Def. surgery along g is the map $S^{\deg \Gamma} \to B\,\mathrm{Diff}_\partial X$ obtained as the result of surgery on $L_{g,t} \subseteq X$

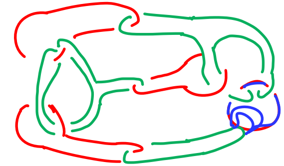
That is: $X_{L_{g,t}} := \left(X \setminus L_{g,t}\left(\bigsqcup_{\# E} S^1 \times D^3 \sqcup \bigsqcup_{\# E} S^2 \times D^2\right)\right) \underset{\text{framing}}{\cup} \left(\bigsqcup_{\# E} D^2 \times S^2\right) \sqcup \left(\bigsqcup_{\# E} D^3 \times S^1\right)$

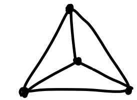
! fibre bundles over a space Y with fibre X and structure group $G \leq \mathrm{Homeo}(X)$ are classified by $[Y, BG]$
examples: $SO(4) \leq O(4) \leq \mathrm{Diffeo}(\mathbb{R}^4) \leq \mathrm{Homeo}(\mathbb{R}^4)$ for bundles with fibre $X = \mathbb{R}^4$.

! think of a point $P \in B\mathrm{Diff}(X)$ as a manifold $(\subseteq \mathbb{R}^\infty)$ that is diffeomorphic to X (but a choice unspecified)

§ examples.

$\Theta =$ 



$K_4 =$  (exercise)

§ THM [Watanabe '19, '22] For $X = S^4$ we have:

$$\begin{tikzcd}
\pi_n B\mathrm{Diff}(X) \arrow[rrr, "② \ \exists Z_n"] & & & \mathcal{A}_n^{even} \otimes \mathbb{Q} \\
\mathbb{Z}[\mathrm{Graph}_n^{even}] \arrow[u, "R_n"']
\end{tikzcd}$$

① surj. homom. of abelian groups

$\mathrm{wat}(g) \longmapsto Z_n(U_g) = [\Gamma]$

$\uparrow$

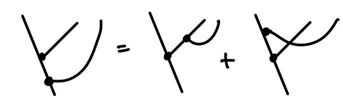
$[\Gamma]$ ③

Here: $\mathcal{A}_n^{even} := \dfrac{\mathbb{Z}[\mathrm{Graph}_n^{even}]}{AS, IHX}$

Z_n constructed using integration over (compactified) config. spaces

Cor. $\pi_n B\mathrm{Diff}_\partial(X) \otimes \mathbb{Q} \twoheadrightarrow \mathcal{A}_n^{even} \otimes \mathbb{Q}$.

AS: if Γ' is same as Γ except has edge order permuted by an odd permutation: $\Rightarrow \Gamma' = -\Gamma$

IHX: 

Thm [Botvinnik-Watanabe '23] $\exists A_{g,t} : S^1 \hookrightarrow X \# S^1 \times S^3$ $\forall t \in S^n$, isotopic to $S^1 \times pt$

s.t. $\mathrm{wat}(g) : S^n \to B\mathrm{Diff}(X)$ is homotopic to

$$S^n \to B\mathrm{Diff}_{\partial_0}(X \times I) \xrightarrow{\partial_1} B\mathrm{Diff}(X)$$