~ ICTP Summer School ~

Lecture 2

Kontsevich invariants
& configuration spaces

June 2, 2025

II

- for a topological group G there is a classifying space BG, s.t. fibre bundles over a space Y with fibre X and structure group $G \longrightarrow \mathrm{Homeo}(X)$ up to iso are classified by $[Y, BG] = \mathrm{Map}(Y, BG)/\mathrm{htpy}$.
- examples. $\mathrm{Spin}(4) \longrightarrow SO(4) \hookrightarrow O(4) \hookrightarrow \mathrm{Diff}(\mathbb{R}^4) \hookrightarrow \mathrm{Homeo}(\mathbb{R}^4)$ for fibre $X = \mathbb{R}^4$
- $\pi_n BG \cong \pi_{n-1} G$. exercise.
- think of a point in $B\mathrm{Diff}(X)$ as a manifold ($\subseteq \mathbb{R}^\infty$) that is diffeomorphic to X (no choice fixed)

Recall: Γ = trivalent graph of $\deg \Gamma = n$ $\left(= \#V/2 = \#E/3\right)$ $\quad g : \Gamma \hookrightarrow X \rightsquigarrow L_{g,\vec{t}} : \coprod_{3n} S^1 \sqcup \coprod_{3n} S^2 \longrightarrow X$, $\quad \vec{t} \in S^n = [0,1]^n/\sim$ (n circles move)

Def. surgery along g is the map $S^{\deg\Gamma} \to B\mathrm{Diff}_\partial X$ obtained as the result of surgery on $L_{g,\vec{t}} \subseteq X$ $\quad \forall\, t_i \in [0,1]$

$$S^{\deg\Gamma} = [0,1]^{\deg\Gamma}/\sim$$

$$\vec{t} = (t_1, \dots, t_n) \mapsto X_{L_{g,\vec{t}}}$$

note: $X_{L_{g,\vec{t}}} \cong X$ because 

§ THM [Watanabe '19, '22] For $X = S^4$ we have:

$$\begin{tikzcd} \pi_n BDiff(X) \arrow[rr, "② \ \exists Z_n"] & & \mathcal{A}_n^{even} \otimes \mathbb{Q} \\ \mathbb{Z}[Graph_n^{even}] \arrow[u, "\mathcal{R}_n"'] & & \end{tikzcd}$$

① surj. homom. of abelian groups

$wat(\mathcal{G}) \longmapsto [\Gamma]$

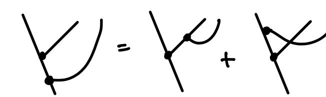
$[\Gamma] \mapsto wat(\mathcal{G})$ ③

Here: $\mathcal{A}_n^{even} := \dfrac{\mathbb{Z}[Graph_n^{even}]}{AS, IHX}$

Z_n constructed using integration over (compactified) config. spaces

AS: if Γ' is same as Γ except has edge order permuted by an odd permutation: $\Rightarrow \Gamma' = -\Gamma$

IHX:



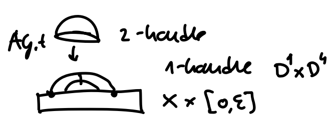
Cor. $\pi_n BDiff_\partial(X) \otimes \mathbb{Q} \twoheadrightarrow \mathcal{A}_n^{even} \otimes \mathbb{Q}$.

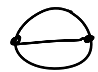
✧ In fact: the 2#E component link can be reduced to 2-components.

Thm [Botvinnik-Watanabe '23] $\exists\, A_{\mathcal{G},\vec{t}} : S^1 \hookrightarrow X \# S^1 \times S^3 \quad \forall \vec{t} \in S^n$, isotopic to $S^1 \times pt$

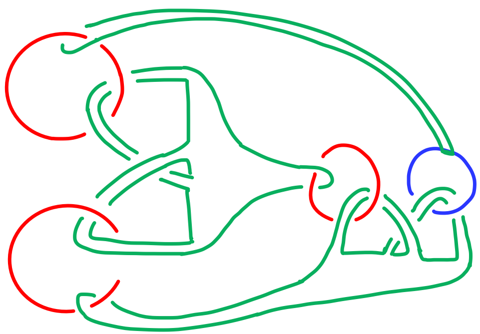
s.t. $wat(\mathcal{G}) : S^n \to BDiff(X)$ is homotopic to

$$S^n \longrightarrow BDiff_{\partial_0}(X \times I) \xrightarrow{\partial_1} BDiff(X)$$

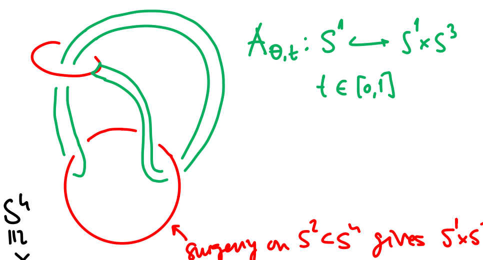


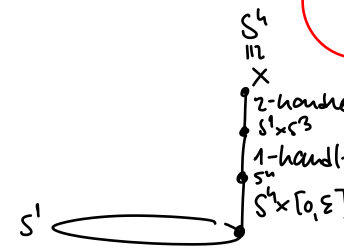
now consider only $n=1$.  $\longmapsto \ \mathrm{wat}(\Theta) \in \pi_1 B\mathrm{Diff}(X) \xrightarrow{Z_1} \mathcal{A}_1 \otimes \mathbb{Q} = 0$

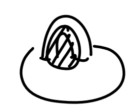
$$\pi_1 B\mathrm{Diff}(X) \cong \pi_0 \mathrm{Diff}(X), \qquad \mathcal{A}_1 = \frac{\mathbb{Z}[\Theta]}{\Theta = -\Theta} = \mathbb{Z}/2$$



cancel handles $\rightsquigarrow$







$\Longrightarrow$ $\mathrm{wat}(\Theta) \in \pi_1 B\mathrm{Diff}(S^4)$ is the top of

$\cong$

$\pi_0 \mathrm{Diff}(S^4)$ is the monodromy of that bundle

✧ How to see it explicitly? AMBIENT ISOTOPY EXTENSION 

start going around the circle and identifying all fibres with X

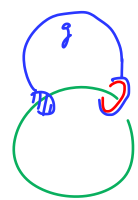
$\longleftrightarrow$ ambiently extend $A_{\Theta,t}$ to $\Phi_t: S^1 \times S^3 \to S^1 \times S^3$ s.t. $\Phi_t(S^1 \times pt) = A_t$

$\Phi_0 = \mathrm{Id}$

Thm [K '24] In any X^4 Watanabe's class $\mathrm{wat}(g)$ for any $g: \ominus \hookrightarrow X$ is in the image of the map

ps: $\pi_1\left(\mathrm{Emb}^{fr}(S^1, X \# S^1\times S^3);\ S^1\times D^3\right) \xrightarrow{\mathrm{amb}} \pi_0 \mathrm{Diff}_\partial (X \# S^1\times D^3) \xrightarrow{\cup \mathrm{Id}_{D^2\times S^2}} \pi_0 \mathrm{Diff}_\partial(X)$

classes here



$g \in \pi_1(X \# S^1\times S^3)$

◇ [Budney - Gabai '19] $\mathbb{Z}^\infty < \pi_0 \mathrm{Diff}_\partial(S^1\times S^3)$ also in im(ps), for $X = S^1\times S^3$.

◇ Q: Is $\mathrm{wat}(\Theta) \in \pi_0 \mathrm{Diff}(S^4)$ non-trivial?

T [Gay' Gay-Hartman] $\mathrm{wat}(\Theta)^2 \simeq \mathrm{Id}$

Rem. Gabai-Gay-Hartman have announced $\mathrm{wat}(\Theta) \not\simeq \mathrm{Id}$.

In fact, both of these results use that $\mathrm{wat}(\Theta) = \mathrm{bg}(\ominus)$ [Gay, K.]