KNOT INVARIANTS FROM HOMOTOPY THEORY

Embedding calculus and grope cobordism of knots

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Based on [Kos20].

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Introduction

Spaces of embeddings

Goal. Study the homotopy type of the space

$$\mathrm{Emb}_\partial(X, M)$$

of smooth neat embeddings between compact manifolds X and M, satisfying a fixed boundary condition $\partial X \hookrightarrow \partial M$.

Remarks.

- Neat embedding is the one that is transverse to ∂M.
- We use Whitney C^∞-topology.
- The case of closed manifolds can be reduced to this.

Tools. Goodwillie-Weiss [GW99] embedding calculus

$\Rightarrow$ homotopy limits, Whitehead / Samelson products, configuration spaces, operads, graph complexes...

In this talk...

...we give a geometric interpretation of embedding calculus for

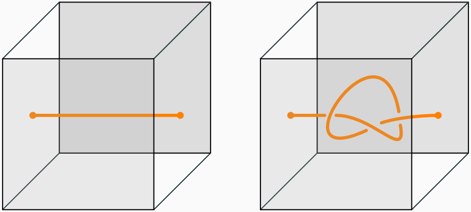
$$\mathrm{Emb}_\partial(I, M) \quad \text{with} \quad I = [0,1] \text{ and } \quad \dim(M) = 3$$

and relate it to Vassiliev theory of finite type knot invariants [Vas90].

In particular, for $M = I^3$ one has

$$\pi_0 \, \mathrm{Emb}(\mathbb{S}^1, \mathbb{S}^3) \cong \pi_0 \, \mathrm{Emb}_\partial(I, I^3) = {\{\text{knots}\}}/_{\text{isotopy}}$$

which is a commutative monoid:



Geometric approach to embedding calculus

Embedding calculus: the Taylor tower

The outcome of this theory is the tower:

Theorem (Goodwillie-Klein [GK15])

If $(\dim X, \dim M) \neq (1,3)$ *then the map* ev_n *is* k*-connected for*

$$k := \big(1 - \dim X + n(\dim M - \dim X - 2)\big).$$

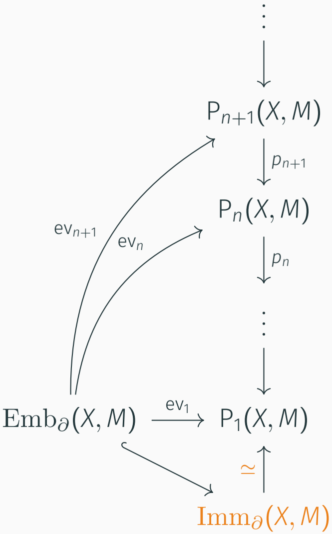
Recall: a map is k-connected if it is an iso on $\pi_{*<k}$ and onto π_k.

Corollary

If $\dim M - \dim X > 2$*, then*

$$\lim \mathrm{ev}_n : \operatorname{Emb}_\partial(X, M) \to \lim \mathrm{P}_n(X, M)$$

is a weak equivalence.



The Taylor tower for knotted arcs

Note. One can show that $\lim \mathrm{ev}_n$ for $M = I^3$ is not a weak equivalence. However the connectivity formula predicts:

If $\dim M = 3$, then $\mathrm{ev}_n \colon \mathrm{Emb}_\partial(I, M) \to \mathrm{P}_n(I, M)$ is 0-connected.

This follows from Main Theorem B (see p. 9), which gives a geometric interpretation of points in $\mathrm{P}_n(I, M)$. We use the following model.

Fix an increasing sequence of disjoint closed subintervals $J_i \subseteq I$.

Definition (Goodwillie's punctured knots model)

$$\mathrm{P}_n(I, M) := \underset{\emptyset \neq S \subseteq \{0,1,\dots,n\}}{\mathrm{holim}} \mathrm{Emb}_\partial(I \setminus \bigsqcup_{i \in S} J_i, M)$$

There are natural fibrations p_n and evaluation maps ev_n.

Connection to Vassiliev's theory

Theorem (Budney-Conant-Koytcheff-Sinha [BCKS17])

The set $\pi_0 \mathsf{P}_n(I, I^3)$ has a structure of an abelain group, and $\pi_0 \mathsf{ev}_n \colon \pi_0 \operatorname{Emb}_\partial(I, I^3) \to \pi_0 \mathsf{P}_n(I, I^3)$ is an additive Vassiliev invariant of type $< n$, that is, a map of monoids which factors as

$$\begin{array}{ccc} \pi_0 \operatorname{Emb}_\partial(I, I^3) & \xrightarrow{\pi_0 \mathsf{ev}_n} & \pi_0 \mathsf{P}_n(I, I^3) \\ \downarrow & \nearrow_{\overline{\mathsf{ev}}_n} & \\ \pi_0 \operatorname{Emb}_\partial(I, I^3) / \sim_n & & \end{array}$$

Conjecture (BC-Scannell-S [BCSS05])

$\pi_0 \mathsf{ev}_n$ is a universal additive Vassiliev invariant of type $< n$, that is, the induced homomorphism $\overline{\mathsf{ev}}_n$ is an isomorphism of groups.

Corollary (of Theorem A)

The homomorphism $\overline{\mathsf{ev}}_n$ is surjective.

Connection to Vassiliev's theory

Moreover, we can combine our main theorem and results of Boavida de Brito and Horel to obtain the following.

Corollary (of Theorem B and [BH20])

- $\overline{\mathrm{ev}}_n \otimes \mathbb{Q}$ is an isomorphism for all $n \geq 1$.
- $\overline{\mathrm{ev}}_n \otimes \mathbb{Z}_p$ is an isomorphism for $n \leq p + 2$.

Remarks.

- Thus, the embedding calculus invariants are at least as good as Kontsevich integral (or Bott–Taubes configuration space integrals).
- Those invariants indeed use integration, so cannot offer answers over $\mathbb{Z}$ (or in characteristic p).
- As a consequence of Theorem B, we also have that they factor through the Taylor tower, cf. [Vol06].

Two disguises of trees

There is a geometric approach to Vassiliev's theory using gropes.

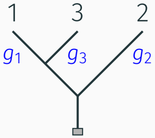
A grope cobordism of degree n is a certain 2-complex built out of surfaces which are embedded in M. It has an underlying $\pi_1 M$-decorated tree $\Gamma^{g_{\underline{n}}} \in \mathrm{Tree}_{\pi_1 M}(n)$.

Here $\Gamma^{g_{\underline{n}}}$ consists of a rooted planar binary tree Γ with n leaves which are enumerated and also decorated by elements

$$g_i \in \pi_1(M), \quad 1 \leq i \leq n.$$

For example,

$\Gamma^{g_{\underline{3}}} :=$



$\in \mathrm{Tree}_{\pi_1 M}(3).$

Remarkably, the first non-vanishing homotopy group of the layer

$$\mathcal{F}_{n+1}(M) := \mathrm{fib}\Big(p_{n+1}\colon \mathsf{P}_{n+1}(I,M) \to \mathsf{P}_n(I,M)\Big)$$

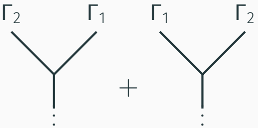
in the Taylor tower is also related to the set $\mathsf{Tree}_{\pi_1 M}(n)$.

Namely, for any $\mathsf{dim}(M) = d \geq 3$ we show that

$$\pi_{n(d-3)}\mathcal{F}_{n+1}(M) \cong \mathsf{Lie}_{\pi_1 M}(n) := {}^{\mathbb{Z}[\mathsf{Tree}_{\pi_1 M}(n)]}\!\big/_{AS,\, IHX} \cong \mathsf{Lie}(n) \otimes \mathbb{Z}[(\pi_1 M)^n]$$

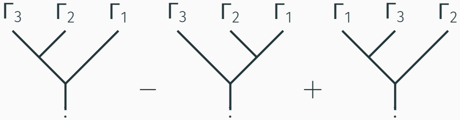
where

AS :



$= 0,$

IHX :



$= 0.$

Two disguises of trees are compatible

Main Theorem B [Kos20]

Given a grope cobordism $\mathcal{G}$ of degree n there is a point

$$\mathrm{e}_{n+1}\psi(\mathcal{G}) \in \mathcal{F}_{n+1}(M)$$

and its path component is precisely given by the class modulo AS, IHX of the underlying tree

$$\mathrm{t}(\mathcal{G}) \in \mathrm{Tree}_{\pi_1 M}(n).$$

Remark

Linear combinations of trees are realised by "grope forests" (higher genus gropes) and the appropriate extension of the theorem holds.

See p. 16 for maps e_{n+1} and ψ.

More details

Finite type knot invariants

Vassiliev '90 studied a stratification of $Map(I, I^3) \setminus \mathrm{Emb}_\partial(I, I^3)$.

Definition

A knot invariant $v\colon \pi_0 \mathrm{Emb}_\partial(I, I^3) \to A$ is of type $< n$ if its natural extension to knots with $< n$ double points vanishes.

Kontsevich '91 defined a universal (additive) invariant of type $< n$

$$Z^{Kont}_{<n}\colon \pi_0 \mathrm{Emb}_\partial(I, I^3) \to \prod_{k<n} \mathcal{A}^t_k \otimes \mathbb{Q}$$

where $\mathcal{A}^t_k := {}^{\mathsf{Lie}(k)}\!\big/_{STU^2}$ is the group of Jacobi trees. Namely, any type $< n$ invariant over $\mathbb{Q}$ factors through this one.

Example. All quantum invariants are of finite type, and can be written as $\omega \circ Z^{Kont}_{<n}$ using weight systems $\omega_k\colon \mathcal{A}^t_k \to \mathbb{C}$.

Question

What is a geometric meaning of Jacobi trees?

Geometric approach to finite type theory

Theorem (Gusarov [Gus00] Habiro [Hab00] Conant-Teichner [CT04])

Two knots $K_0, K_1 \colon I \hookrightarrow I^3$ have the same invariants of type $< n$ if and only if there is a sequence of (capped) grope cobordisms of degree n from K_0 to K_1. We write $K_0 \sim_n K_1$.

Think of a grope cobordism as an ambient cobordism from a subset of K_0 to a subset of K_1, which has several layers of embedded surfaces following a shape of a tree. *Examples follow shortly.*

Actually, we can define this in any 3-manifold M, and there is the underlying tree map t_n that fits into

$$\begin{array}{ccccc}
\pi_0 \operatorname{Emb}_\partial(I, M) & \xleftarrow{\ \partial_0\ } & \pi_0 \mathrm{Grop}_n(M) & \xrightarrow{\ \mathrm{t}_n\ } & \mathbb{Z}[\mathrm{Tree}_{\pi_1 M}(n)] \\
\big\downarrow & & & & \big\downarrow {\scriptstyle AS, IHX, STU^2} \\
\pi_0 \operatorname{Emb}_\partial(I, M)/\sim_{n+1} & \underset{\mathfrak{R}_n}{\xleftarrow{\quad\quad\quad\quad\quad}} & & \xrightarrow{Z^{Kont} \text{ over } \mathbb{Q} \text{ for } M = I^3} & \mathcal{A}^t_n(M)
\end{array}$$

Examples: grope cobordisms of degree 1

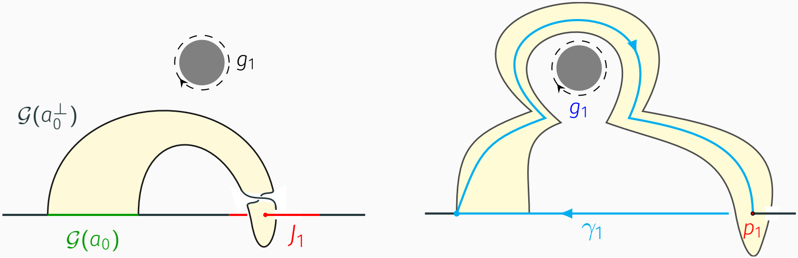
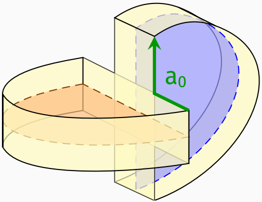


Figure 1: Two grope cobordisms of degree 1 on $K_0 \colon I \hookrightarrow M$ (the horizontal line). In both cases the union of black and red arcs is

$$K_1 := \big(\mathrm{U} \setminus \mathcal{G}(a_0)\big) \cup \mathcal{G}(a_0^\perp)$$

The trees are given by

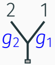
Examples: grope cobordisms of degree 2



Right: A grope cobordism from U to the knot

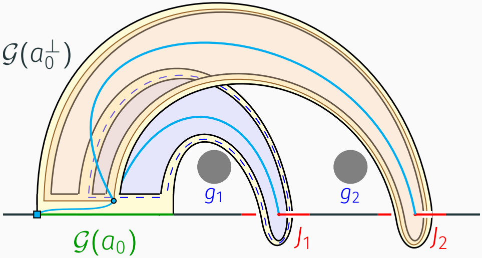
$$\partial^\perp \mathcal{G} = \Big(\mathrm{U}\setminus\mathcal{G}(a_0)\Big)\cup\mathcal{G}(a_0^\perp).$$

with underlying tree:

$\mathfrak{t}(\mathcal{G}) =$ 

Left: Abstract grope G_Γ for $\Gamma =$ 

is the union of the yellow torus and two disks.



$\mathcal{G}\colon G_\Gamma \to I^3$

Examples: n-equivalence of knots

Recall: $K_0 \sim_n K_1$ if there is a sequence of (capped) grope cobordisms of degree n from K_0 to K_1. For example:

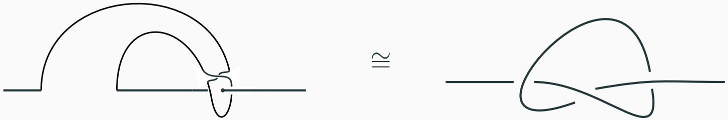


Figure 1: The right-handed trefoil *RHT* is 1-equivalent to the unknot.

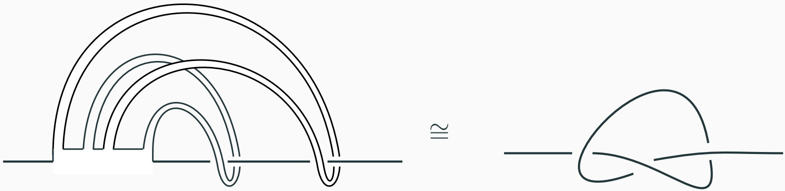


Figure 2: *RHT* is 2-equivalent to the unknot. But $RHT \not\sim_3 \mathrm{U}$.

Gropes give points in the Taylor layers

Theorem K-Shi-Teichner

There is a continuous map ψ so that the following commutes

$$\begin{array}{ccccc}
\mathrm{Grop}_n(M;\mathrm{U}) & \xrightarrow{\psi} & \mathcal{H}_n(M) := \mathrm{hofib}(\mathrm{ev}_n) & \overset{\mathrm{e}_{n+1}}{\dashrightarrow} & \mathcal{F}_{n+1}(M) \\
& \searrow^{\partial_1} & \downarrow & & \downarrow \\
& & \mathrm{Emb}_\partial(I,M) & \xrightarrow{\mathrm{ev}_{n+1}} & \mathrm{P}_{n+1}(M) \\
& & \downarrow \mathrm{ev}_n & & \downarrow p_n \\
& & \mathrm{P}_n(M) & = & \mathrm{P}_n(M)
\end{array}$$

Theorem [Kos20]

For any M of dimension $d \geq 3$ there is a homotopy equivalence

$$\mathcal{F}_{n+1}(M) \simeq \Omega^{n+1} \prod_{w \in BL(n)} \Sigma^{1+l_w(d-2)}(\Omega M^{\times l_w})_+$$

and the first non-vanishing homotopy group is

$$\pi_{n(d-3)}\mathcal{F}_{n+1}(M) \cong \mathrm{Lie}_{\pi_1 M}(n).$$

Main theorem more precisely

Main Theorem B [Kos20]

The following diagram commutes

$$\begin{array}{ccc}
\pi_0 \mathrm{Grop}_n(M) & \xrightarrow{\ \mathrm{t}_n\ } & \mathbb{Z}[\mathrm{Tree}_{\pi_1 M}(n)] \\
\psi \big\downarrow & & \big\downarrow AS, IHX \\
\pi_0 \mathcal{H}_n(M) & \overset{\pi_0 \mathrm{e}_{n+1}}{\dashrightarrow} & \mathrm{Lie}_{\pi_1 M}(n)
\end{array}$$

Corollary

The map $\pi_0 \mathrm{e}_{n+1}$ is a surjection (of sets).

- Theorem A (that $\pi_0 \mathrm{ev}_n$ is onto) follows from this by induction.
- Corollaries about universality follow using the work of Conant [Con08] and by considering the spectral sequence in homotopy groups of the tower of fibrations p_n.

Thank you!

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